

HIERARCHICAL MEAN-FIELD THEORIES

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1. INTRODUCTION TO COMPLEX QUANTUM ORDERS

One of the most challenging and interesting phenomena in modern condensed matter physics is the one emerging from strongly coupled systems as a result of competing interactions. Indeed, the multiplicity of distinct and novel quantum phases observed experimentally confront us with new paradigms that question our understanding of the fundamental principles behind such complex phenomena. For example, whether the mechanism controlling the coexistence and/or competition between magnetism and superconductivity or Bose-Einstein condensation has the same physical origin in different classes of materials is still an open question [1]. It is believed that the physics in these materials is strongly influenced by their proximity to quantum phase transitions and, in particular, to quantum criticality. A quantum phase transition is characterized by the qualitative changes of the macroscopic state of the system induced by tuning parameters of its Hamiltonian. On general grounds, the very notion and nature of entanglement is at the core of the problem [2].

From the theoretical viewpoint the hurdle is in the presence of non-linear couplings, non-perturbative phenomena, and a panoply of competing quantum orders. These systems happened to be strongly correlated since no obvious small coupling constant exists, and consequently exhibit high sensitivity to small parameter changes. It is then clear the importance of developing a methodology, based on qualitatively new concepts, that treats all possible competing orders on an equal footing with no privileged fixed-point phenomenon. Despite great advances there is a lack of a systematic and reliable methodology to study and predict the behavior of these complex systems. It is a purpose of this work to present a promising step in that direction.

In the quantum description of matter, a physical system is naturally associated with a *language* of operators. We have previously developed an algebraic framework for interacting quantum systems that let us study complex phenomena characterized by the coexistence and competition of various broken symmetry states [3-5], and proved a theorem that allowed us to connect all possible languages used in the quantum description of matter. Connecting the various languages through isomorphic

mappings enable us to relate seemingly different physical phenomena, unveil hidden symmetries (i.e., uncover the *accidental degeneracies* of the original physical system), and, in some limiting cases, obtain the exact spectrum of the problem (or of a set of orthogonal subspaces). The ultimate goal was to use that framework to explore those unconventional complex states of matter from a unified perspective. Given the space limitations we cannot review these concepts but refer the reader to a recent review article on the subject [5].

The modern theory of phase transitions starts with Landau's pioneering work in 1937 [6]. One of his achievements was the realization of the fundamental relation between spontaneous symmetry breaking and the order parameter (OP) that measures this violation, thus giving simple prescriptions to describe order in terms of irreducible representations of the symmetry group involved. Another was the development of a phenomenological calculational scheme to study the behavior of systems near a phase transition. Landau's theory has been successfully applied to study phase transitions where thermal fluctuations are most relevant. Certainly, the theory was not designed to study zero-temperature (quantum) phase transitions.

In previous work [3,4] we outlined a framework to identify OPs based upon isomorphic mappings to a *hierarchical language* (HL) defined by the set of operators which in the fundamental representation (of dimension D) has the largest number of symmetry generators of the group. *Any* local operator can be expressed as a *linear* combination of the generators of the HL. The building of the HL depends upon the dimension D of the local Hilbert space, \mathcal{H}_j , modeling the physical phenomena. For instance, if one is modeling a doped antiferromagnetic (AF) insulator with a t - J Hamiltonian [7], then $D=3$ (i.e., there are three possible states per site) and a HL is generated by a basis of the Lie algebra $su(3)$ in the fundamental representation [3,4]. As explained and proved in Refs. [3-5], there is always a HL associated to each physical problem. These ideas complement Landau's concept of an OP providing a mechanism to reveal them, something that is outside the groundwork of his theory. Indeed, Landau's theory does not say what the OPs should be in a general situation.

As mentioned above, these isomorphic mappings not only unveil hidden symmetries of the original physical system but also manifestly establish equivalences between seemingly unrelated physical phenomena. Nonetheless, this is not sufficient to determine the *exact* phase diagram of the problem: One has to resort to either numerical simulations with their well-known limitations or, as will be shown in the present paper, to a *guided* approximation which at least preserves the qualitative nature of the possible thermodynamic states. A key observation in this regard is the fact that typical model Hamiltonian operators written in the HL become quadratic in the symmetry generators of the hierarchical group, and this result is independent of the group of symmetries of the Hamiltonian.

This latter result suggests a simple approximation, based upon group theoretical grounds, which deals with competing orders on an equal footing and will be termed *hierarchical mean-field theory* (HMFT). The HMFT is distinctly suitable when the various phases displayed by a system are the result of competing interactions and non-linear couplings of their constituents matter fields.

2. HIERARCHICAL MEAN-FIELD THEORY AT WORK

In the rest of the paper we will expand on a technique to build approximate phase diagrams, dubbed HMFT. To avoid excess of tedious formalism we will describe the methodology by example. We have chosen two representative examples. The first displays two quantum phases separated by a bosonic *metal-insulator* transition induced by particle interactions. The second shows an AF, a superfluid and, in addition, a coexistence phase as a function of the particle density and temperature.

2.1 A Superfluid-Insulator Transition

Let us determine the zero temperature phase diagram of a very simple model displaying a Mott insulating to superfluid transition. The model we refer to is a modification of the Bose-Hubbard Hamiltonian [8] ($U, t > 0$)

$$\begin{aligned} H &= -2t \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} (g_{\mathbf{i}}^\dagger g_{\mathbf{j}} + g_{\mathbf{j}}^\dagger g_{\mathbf{i}}) + U \sum_{\mathbf{j}} (n_{\mathbf{j}} - \bar{n} + 1)(n_{\mathbf{j}} - \bar{n}) + \sum_{\mathbf{j}} \epsilon_{\mathbf{j}} n_{\mathbf{j}} , \\ &= -2t \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} (g_{\mathbf{i}}^\dagger g_{\mathbf{j}} + g_{\mathbf{j}}^\dagger g_{\mathbf{i}}) + U \sum_{\mathbf{j}} n_{\mathbf{j}}(n_{\mathbf{j}} - 1) + \sum_{\mathbf{j}} \mu_{\mathbf{j}} n_{\mathbf{j}} + N_s U \bar{n}(\bar{n} - 1) , \end{aligned} \quad (1)$$

where $\langle \mathbf{i}, \mathbf{j} \rangle$ stands for nearest-neighbor sites (bond) in an otherwise regular N_s -sites lattice of coordination z , $\mu_{\mathbf{j}} = 2U(1 - \bar{n}) + \epsilon_{\mathbf{j}}$, and the g -particles, instead of canonical bosons, represent bosons satisfying the following commutation relations $((g_{\mathbf{j}}^\dagger)^\dagger = g_{\mathbf{j}}, n_{\mathbf{j}} \neq g_{\mathbf{j}}^\dagger g_{\mathbf{j}})$

$$\begin{cases} [g_{\mathbf{i}}, g_{\mathbf{j}}] = [g_{\mathbf{i}}^\dagger, g_{\mathbf{j}}^\dagger] = 0 , \\ [g_{\mathbf{i}}, g_{\mathbf{j}}^\dagger] = \delta_{\mathbf{i}\mathbf{j}} \bar{n}(\bar{n} - n_{\mathbf{j}}) , \quad [n_{\mathbf{i}}, g_{\mathbf{j}}^\dagger] = \delta_{\mathbf{i}\mathbf{j}} g_{\mathbf{j}}^\dagger , \end{cases} \quad (2)$$

defining an algebra, with $\bar{n} \geq 1$ a positive integer and the nilpotency condition $(g_{\mathbf{j}}^\dagger)^{p+1} = (g_{\mathbf{j}}^\dagger)^3 = 0$, meaning that one can accommodate up to $p = 2$ particles per mode \mathbf{j} . This indicates that the local Hilbert space $\mathcal{H}_{\mathbf{j}}$ is three-dimensional. A possible basis is $\{|\bar{n} - 1\rangle, |\bar{n}\rangle, |\bar{n} + 1\rangle\}$, and the operators acting on this basis

$$\begin{aligned} g_{\mathbf{j}}|\bar{n} - 1\rangle &= 0 , & g_{\mathbf{j}}^\dagger|\bar{n} - 1\rangle &= \sqrt{\bar{n}} |\bar{n}\rangle , & n_{\mathbf{j}}|\bar{n} - 1\rangle &= (\bar{n} - 1)|\bar{n} - 1\rangle , \\ g_{\mathbf{j}}|\bar{n}\rangle &= \sqrt{\bar{n}} |\bar{n} - 1\rangle , & g_{\mathbf{j}}^\dagger|\bar{n}\rangle &= \sqrt{\bar{n}} |\bar{n} + 1\rangle , & n_{\mathbf{j}}|\bar{n}\rangle &= \bar{n}|\bar{n}\rangle , \\ g_{\mathbf{j}}|\bar{n} + 1\rangle &= \sqrt{\bar{n}} |\bar{n}\rangle , & g_{\mathbf{j}}^\dagger|\bar{n} + 1\rangle &= 0 , & n_{\mathbf{j}}|\bar{n} + 1\rangle &= (\bar{n} + 1)|\bar{n} + 1\rangle , \end{aligned} \quad (3)$$

behave as harmonic-oscillator-like operators acting on a three-dimensional space. Since $D = 3$ one can easily determine the isomorphic mapping between the g -particle and the $su(2)$ algebra in the $S = 1$ representation. The transformation is given by

$$\begin{cases} \sqrt{\frac{2}{\bar{n}}} g_{\mathbf{j}}^\dagger = S_{\mathbf{j}}^+ , \\ \sqrt{\frac{2}{\bar{n}}} g_{\mathbf{j}} = S_{\mathbf{j}}^- , \\ n_{\mathbf{j}} - \bar{n} = S_{\mathbf{j}}^z , \end{cases} \quad (5)$$

while the Hamiltonian operator in the spin language reads

$$H = -t\bar{n} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} (S_{\mathbf{i}}^+ S_{\mathbf{j}}^- + S_{\mathbf{i}}^- S_{\mathbf{j}}^+) + U \sum_{\mathbf{j}} (S_{\mathbf{j}}^z)^2 + \sum_{\mathbf{j}} (U + \epsilon_{\mathbf{j}}) S_{\mathbf{j}}^z . \quad (6)$$

Before translating the problem into the most fundamental language, i.e., the HL, let us make a detour and re-express H using the spin-1/2 ($\sigma = \uparrow, \downarrow$) Jordan-Wigner bosons [9],

$$\begin{cases} S_j^+ = \sqrt{2} (\bar{b}_{j\uparrow}^\dagger + \bar{b}_{j\downarrow}) , \\ S_j^- = \sqrt{2} (\bar{b}_{j\uparrow} + \bar{b}_{j\downarrow}^\dagger) , \\ S_j^z = \bar{n}_{j\uparrow} - \bar{n}_{j\downarrow} , \end{cases} \quad (7)$$

$$H = -2t\bar{n} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle, \sigma} (\bar{b}_{i\sigma}^\dagger \bar{b}_{j\sigma} + \bar{b}_{i\sigma}^\dagger \bar{b}_{j\bar{\sigma}}^\dagger + \text{H.c.}) + U \sum_{\mathbf{j}} \bar{n}_{\mathbf{j}} + \sum_{\mathbf{j}} (U + \epsilon_{\mathbf{j}}) (\bar{n}_{j\uparrow} - \bar{n}_{j\downarrow}) , \quad (8)$$

where the number operator $\bar{n}_{\mathbf{j}} = \bar{n}_{j\uparrow} + \bar{n}_{j\downarrow}$ ($\bar{n}_{j\sigma} = \bar{b}_{j\sigma}^\dagger \bar{b}_{j\sigma}$), and the algebra satisfied by these hard-core bosons is [5]: $[\bar{b}_{i\sigma}, \bar{b}_{j\sigma'}] = 0$, $[\bar{b}_{i\sigma}, \bar{b}_{j\sigma'}^\dagger] = \delta_{ij}(1 - 2\bar{n}_{j\sigma} - \bar{n}_{j\bar{\sigma}})$ (if $\sigma = \sigma'$), or $-\delta_{ij}\bar{b}_{j\sigma'}^\dagger \bar{b}_{j\sigma}$ (if $\sigma \neq \sigma'$).

As explained in the introduction, the first step in determining its phase diagram consists of re-writing H in a HL. The latter is realized by $SU(3)$ -spin generators in the fundamental representation, and its mapping to the hard-core boson language can be compactly written as [3]

$$\mathcal{S}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - \bar{n}_{\mathbf{j}} & \bar{b}_{j\uparrow} & \bar{b}_{j\downarrow} \\ \bar{b}_{j\uparrow}^\dagger & \bar{n}_{j\uparrow} - \frac{1}{3} & \bar{b}_{j\uparrow}^\dagger \bar{b}_{j\downarrow} \\ \bar{b}_{j\downarrow}^\dagger & \bar{b}_{j\downarrow}^\dagger \bar{b}_{j\uparrow} & \bar{n}_{j\downarrow} - \frac{1}{3} \end{pmatrix}, \quad \tilde{\mathcal{S}}(\mathbf{j}) = \begin{pmatrix} \frac{2}{3} - \bar{n}_{\mathbf{j}} & -\bar{b}_{j\downarrow}^\dagger & -\bar{b}_{j\uparrow}^\dagger \\ -\bar{b}_{j\downarrow} & \bar{n}_{j\downarrow} - \frac{1}{3} & \bar{b}_{j\uparrow}^\dagger \bar{b}_{j\downarrow} \\ -\bar{b}_{j\uparrow} & \bar{b}_{j\downarrow}^\dagger \bar{b}_{j\uparrow} & \bar{n}_{j\uparrow} - \frac{1}{3} \end{pmatrix}. \quad (9)$$

The three components $s_j^z = (\bar{n}_{j\uparrow} - \bar{n}_{j\downarrow})/2$, $s_j^+ = \bar{b}_{j\uparrow}^\dagger \bar{b}_{j\downarrow}$ and $s_j^- = \bar{b}_{j\downarrow}^\dagger \bar{b}_{j\uparrow}$ generate the spin $su(2)$ subalgebra, i.e., they are the components of the local magnetization. The five additional components correspond to the Bose-Einstein condensate and the charge density wave local OPs. In the HL, H represents a Heisenberg-like Hamiltonian [10] in the presence of an external magnetic field ($J_{\mu\nu} = J_{\nu\mu}$)

$$H = \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} J_{\mu\nu} (\mathcal{S}^{\mu\nu}(\mathbf{i}) \mathcal{S}^{\nu\mu}(\mathbf{j}) - \mathcal{S}^{\mu\nu}(\mathbf{i}) \tilde{\mathcal{S}}^{\nu\mu}(\mathbf{j})) - U \sum_{\mathbf{j}} \mathcal{S}^{00}(\mathbf{j}) + \sum_{\mathbf{j}} (U + \epsilon_{\mathbf{j}}) (\mathcal{S}^{11}(\mathbf{j}) - \mathcal{S}^{22}(\mathbf{j})), \quad (10)$$

with $J_{00} = J_{11} = J_{22} = J_{12} = 0$, $J_{01} = J_{02} = -2t\bar{n}$. Note that through the mapping we transformed an interacting problem into another problem that is quadratic in the basis of the algebra $su(3)$ in the fundamental representation, but which is not necessarily $SU(3)$ symmetric. The idea behind the HMFT is to perform an approximation which deals with all possible local OPs on an equal footing with no privileged *symmetry axes* and, hopefully, retains the qualitative topology of the phase diagram. With H written in the HL one immediately realizes that the simplest HMFT can be achieved if we re-write H in terms of $SU(3)$ Schwinger-Wigner (SW) bosons (3 flavors $\alpha = \downarrow, 0, \uparrow$) [7]. The mapping is expressed as

$$\mathcal{S}(\mathbf{j}) = \begin{pmatrix} n_{j0} - \frac{1}{3} & b_{j0}^\dagger b_{j\uparrow} & b_{j0}^\dagger b_{j\downarrow} \\ b_{j\uparrow}^\dagger b_{j0} & n_{j\uparrow} - \frac{1}{3} & b_{j\uparrow}^\dagger b_{j\downarrow} \\ b_{j\downarrow}^\dagger b_{j0} & b_{j\downarrow}^\dagger b_{j\uparrow} & n_{j\downarrow} - \frac{1}{3} \end{pmatrix}, \quad \tilde{\mathcal{S}}(\mathbf{j}) = \begin{pmatrix} n_{j0} - \frac{1}{3} & -b_{j\downarrow}^\dagger b_{j0} & -b_{j\uparrow}^\dagger b_{j0} \\ -b_{j0}^\dagger b_{j\downarrow} & n_{j\downarrow} - \frac{1}{3} & b_{j\uparrow}^\dagger b_{j\downarrow} \\ -b_{j0}^\dagger b_{j\uparrow} & b_{j\downarrow}^\dagger b_{j\uparrow} & n_{j\uparrow} - \frac{1}{3} \end{pmatrix}, \quad (11)$$

with the SW bosons $b_{j\alpha}^\dagger$ satisfying the constraint $n_{j\downarrow} + n_{j0} + n_{j\uparrow} = 1$. In this way

$$H = -2t\bar{n} \sum_{\langle i,j \rangle, \sigma} (\colon A_{\sigma ij}^\dagger A_{\sigma ij} \colon - B_{\sigma ij}^\dagger B_{\sigma ij}) - U \sum_j n_{j0} + \sum_j (U + \epsilon_j)(n_{j\uparrow} - n_{j\downarrow}) , \quad (12)$$

where $\colon \colon$ denotes normal ordering and

$$\begin{cases} A_{\sigma ij}^\dagger = b_{i\sigma}^\dagger b_{j0} + b_{i0}^\dagger b_{j\bar{\sigma}}, \\ B_{\sigma ij}^\dagger = b_{i\sigma}^\dagger b_{j0}^\dagger - b_{i0}^\dagger b_{j\sigma}^\dagger. \end{cases} \quad (13)$$

If ϵ_j is translationally invariant, then $S^z = \sum_j S_j^z = \sum_j (n_{j\uparrow} - n_{j\downarrow})$ is a constant of motion ($[H, S^z] = 0$). In the following we will only consider the case $S^z = 0$.

Since the $su(N)$ languages provide a complete set of HLs [11], any model Hamiltonian can be written in a similar fashion once we identify the appropriate HL and apply the corresponding SW mapping in the *fundamental representation* (the ordering operators will, of course, have a different meaning and algebraic expressions). The key point is that the Hamiltonian operator in the HL becomes quadratic in the symmetry generators of the hierarchical group ($SU(3)$ in the present case).

The idea behind any MF approximation is to disentangle interaction terms into quadratic ones replacing some of the elementary mode operators by their mean value. The crux of our HMFT is that the approximation is done in the HL where all possible local OPs are treated on an equal footing and the number of operators replaced by their mean value is minimized since the Hamiltonian is quadratic in the symmetry generators. In this way, the information required is minimal. In mathematical terms, given $\mathcal{O}_{ij}^\dagger \mathcal{O}_{ij} = \langle \mathcal{O}_{ij}^\dagger \rangle \mathcal{O}_{ij} + \mathcal{O}_{ij}^\dagger \langle \mathcal{O}_{ij} \rangle - \langle \mathcal{O}_{ij}^\dagger \rangle \langle \mathcal{O}_{ij} \rangle + (\mathcal{O}_{ij}^\dagger - \langle \mathcal{O}_{ij}^\dagger \rangle)(\mathcal{O}_{ij} - \langle \mathcal{O}_{ij} \rangle)$, for an arbitrary bond-operator \mathcal{O}_{ij} , the approximation amounts to neglect the latter fluctuations, i.e., $\mathcal{O}_{ij}^\dagger \mathcal{O}_{ij} \approx \langle \mathcal{O}_{ij}^\dagger \rangle \mathcal{O}_{ij} + \mathcal{O}_{ij}^\dagger \langle \mathcal{O}_{ij} \rangle - \langle \mathcal{O}_{ij}^\dagger \rangle \langle \mathcal{O}_{ij} \rangle$. An important result is that all local OPs are equally treated and, moreover, symmetries of the original Hamiltonian related to the OPs are not broken explicitly in certain limits.

The resulting MF Hamiltonian together with the SW-boson constraint (with Lagrange multiplier λ) $\tilde{H} = H_{MF} + \lambda \sum_j (n_{j\downarrow} + n_{j0} + n_{j\uparrow})$ reads [10]

$$\begin{aligned} \tilde{H} &= -2t\bar{n} \sum_{\langle i,j \rangle, \sigma} [A(A_{\sigma ij}^\dagger + A_{\sigma ij}) - iB(B_{\sigma ij}^\dagger - B_{\sigma ij})] - U \sum_j n_{j0} + \lambda \sum_{j,\alpha} n_{j\alpha} \\ &= \sum_{\mathbf{k} \in \text{BZ}} [\Lambda_A b_{\mathbf{k}S}^\dagger b_{\mathbf{k}0} + \Lambda_B b_{\mathbf{k}S}^\dagger b_{-\mathbf{k}0}^\dagger + \text{H.c.}] + \sum_{\mathbf{k} \in \text{BZ}} [(\lambda - U)n_{\mathbf{k}0} + \lambda(n_{\mathbf{k}S} + n_{\mathbf{k}a})], \end{aligned} \quad (14)$$

where the sum of momenta \mathbf{k} is performed over the first Brillouin zone (BZ), $b_{\mathbf{k}S(a)}^\dagger = \frac{1}{\sqrt{2}}[b_{\mathbf{k}\uparrow}^\dagger \pm b_{\mathbf{k}\downarrow}^\dagger]$, $n_{\mathbf{k}\alpha} = b_{\mathbf{k}\alpha}^\dagger b_{\mathbf{k}\alpha}$ ($b_{\mathbf{k}\alpha}^\dagger$'s represent Fourier transformed modes). $\Lambda_A = -2\sqrt{2} t\bar{n} A \gamma_{\mathbf{k}}$, $\Lambda_B = 2\sqrt{2} t\bar{n} B \tilde{\gamma}_{\mathbf{k}}$, with $\gamma_{\mathbf{k}} = 2 \sum_{\mu=1}^d \cos(k_\mu)$ and $\tilde{\gamma}_{\mathbf{k}} = 2 \sum_{\mu=1}^d \sin(k_\mu)$. The resulting self-consistent MF equations are

$$\begin{cases} A = \frac{\sqrt{2}}{ZN_s} \sum_{\mathbf{k} \in \text{BZ}} \gamma_{\mathbf{k}} \langle b_{\mathbf{k}S}^\dagger b_{\mathbf{k}0}^\dagger \rangle_{MF} , \\ B = \frac{\sqrt{2}}{ZN_s} \sum_{\mathbf{k} \in \text{BZ}} \tilde{\gamma}_{\mathbf{k}} \langle b_{\mathbf{k}S}^\dagger b_{-\mathbf{k}0}^\dagger \rangle_{MF} , \\ 1 = \frac{1}{N_s} \sum_{\mathbf{k} \in \text{BZ}} \sum_{\alpha} \langle n_{\mathbf{k}\alpha} \rangle_{MF} . \end{cases} \quad (15)$$

On the other hand, it can be shown that the case with vanishing B -ordering is a stable solution. In the zero-temperature limit all particles condense into the $\mathbf{k} = 0$ mode. This condensation corresponds to the appearance of a superfluid phase of g -particles (or XY $S=1$ ferromagnetism in the spin language) and it is manifested in the non-zero value of A :

$$A = \frac{1}{4\sqrt{2}} \sqrt{16 - \left(\frac{U}{zt\bar{n}}\right)^2}. \quad (16)$$

This expression can be easily obtained from Eq. (15) considering that at $T = 0$ the only non-zero term of the sum is the $\mathbf{k} = 0$ one (i.e., all the particles are condensed). From (16) one can immediately see that the critical value of U/t for the superfluid-insulator transition is: $U_c/t = 4z\bar{n}$.

2.2 Magnetism and Superfluidity

We study now a simple model which displays coexistence and competition between antiferromagnetism and Bose-Einstein condensation (superfluidity). The model represents a gas of interacting spin-1/2 Jordan-Wigner bosons with Hamiltonian ($t > 0$)

$$H = t \sum_{\langle \mathbf{i}, \mathbf{j} \rangle, \sigma} \left(\bar{b}_{\mathbf{i}\sigma}^\dagger \bar{b}_{\mathbf{j}\sigma} + \text{H.c.} \right) + J \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} (\mathbf{s}_{\mathbf{i}} \cdot \mathbf{s}_{\mathbf{j}} - \frac{\bar{n}_{\mathbf{i}} \bar{n}_{\mathbf{j}}}{4}) + V \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \bar{n}_{\mathbf{i}} \bar{n}_{\mathbf{j}} - \bar{\mu} \sum_{\mathbf{j}} \bar{n}_{\mathbf{j}}, \quad (17)$$

where $\mathbf{s}_{\mathbf{j}} = \frac{1}{2} \bar{b}_{\mathbf{j}\mu}^\dagger \vec{\sigma}_{\mu\nu} \bar{b}_{\mathbf{j}\nu}$ is a $s = \frac{1}{2}$ operator ($\vec{\sigma}$ denoting Pauli matrices). Notice that H is an extended t - J -like model of hard-core bosons instead of constrained fermions [10]. These hard-core bosons could represent three-state atoms, like the ones used in trapped Bose-Einstein condensates, moving in an optical lattice potential. For the sake of simplicity we will only consider the AF, $J > 0$, case.

In the HL, H represents a Heisenberg-like Hamiltonian [9] in the presence of an external magnetic field μ' ($J_{\mu\nu} = J_{\nu\mu}$)

$$H = \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} J_{\mu\nu} \mathcal{S}^{\mu\nu}(\mathbf{i}) \mathcal{S}^{\nu\mu}(\mathbf{j}) - \mu' \sum_{\mathbf{j}} \mathcal{S}^{00}(\mathbf{j}), \quad (18)$$

with $J_{00} = V - J/2$, $J_{01} = J_{02} = t$, $J_{11} = J_{12} = J_{22} = J/2$, and $\mu' = \frac{2}{3}(2V - J/2) - \bar{\mu}$. This HL furnishes the natural framework to analyze the symmetries of the Hamiltonian H . There is always an $SU(2)$ spin symmetry generated by $\mathcal{S}^{11} - \mathcal{S}^{22}$, \mathcal{S}^{12} , and \mathcal{S}^{21} . When $\mu' = 0$ and $V = 2t$, there are five additional generators of symmetries related to the charge degrees of freedom. Moreover, if $J = V = 2t$ there is full $SU(3)$ symmetry. For $\mu' \neq 0$, the only charge symmetry that remains is a $U(1)$ symmetry generated by \mathcal{S}^{00} (conservation of the total charge). In this way the HL, leading to a unique OP from which all possible embedded orderings are derived, provides a unified description of the possible thermodynamic states of the system. Yet, it remains to establish the orderings that survive as a result of tuning the parameters of the Hamiltonian or external variables such as temperature and particle filling.

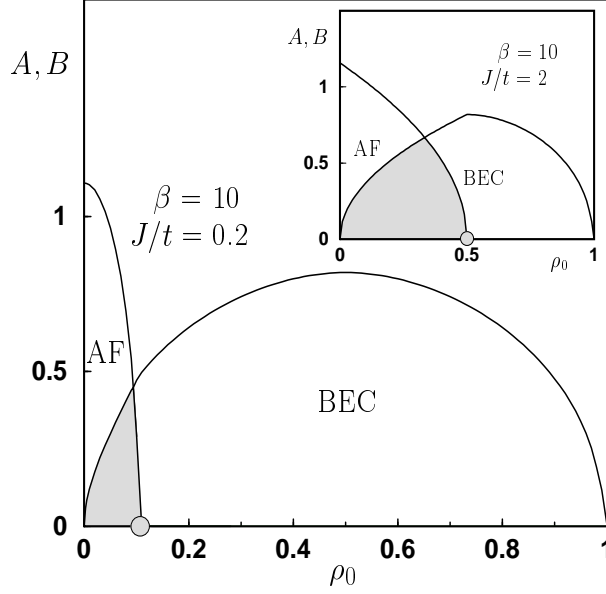


Figure 1. Order fields A and B as a function of the density ρ_0 for different values of J/t and inverse temperature $\beta = 10$ (in units of t^{-1}). The filled circle on the density axis indicates a quantum critical point.

For arbitrary values of the parameters $J/t, V/t$, we do not know a priori how to determine exactly the phase diagram of H [12]. The resulting Hamiltonian ($V = 2t$ with no loss of generality) is [10]

$$H = -\sum_{\langle i,j \rangle} \left(\frac{J}{2} A_{ij}^\dagger A_{ij} + t \sum_{\sigma=\uparrow,\downarrow} B_{\sigma ij}^\dagger B_{\sigma ij} \right) - \mu \sum_{\mathbf{j}} n_{\mathbf{j}0}, \quad (19)$$

where $\mu = zt - \bar{\mu}$ and the ordering operators

$$\begin{cases} A_{ij}^\dagger = b_{i\uparrow}^\dagger b_{j\downarrow}^\dagger - b_{i\downarrow}^\dagger b_{j\uparrow}^\dagger \\ B_{\sigma ij}^\dagger = b_{i\sigma}^\dagger b_{j0}^\dagger - b_{i0}^\dagger b_{j\sigma}^\dagger \end{cases}, \quad (20)$$

which transform as singlets with respect to the generators of $SU(2)$ spin and charge symmetries, respectively: $[A_{ij}^\dagger, \mathcal{S}^{12(21)}(\mathbf{i}) + \mathcal{S}^{12(21)}(\mathbf{j})] = 0 = [B_{\uparrow(\downarrow)ij}^\dagger, \mathcal{S}^{10(20)}(\mathbf{i}) + \mathcal{S}^{10(20)}(\mathbf{j})]$.

The resulting MF Hamiltonian reads [10]

$$\begin{aligned} \tilde{H} &= -\sum_{\langle i,j \rangle} \left[\frac{JA}{2} (A_{ij}^\dagger + A_{ij}) + tB \sum_{\sigma=\uparrow,\downarrow} (B_{\sigma ij}^\dagger + B_{\sigma ij}) \right] - \mu \sum_{\mathbf{j}} n_{\mathbf{j}0} + \lambda \sum_{\mathbf{j},\alpha} n_{\mathbf{j}\alpha} \\ &= \sum_{\mathbf{k} \in \text{RBZ}} [\Lambda_A b_{\mathbf{k}\uparrow}^\dagger b_{-\mathbf{k}+\mathbf{Q}\downarrow}^\dagger + \Lambda_B \sum_{\sigma=\uparrow,\downarrow} b_{\mathbf{k}\sigma}^\dagger b_{-\mathbf{k}+\mathbf{Q}0}^\dagger + \text{H.c.} + (\lambda - \mu) n_{\mathbf{k}0} + \lambda \sum_{\sigma=\uparrow,\downarrow} n_{\mathbf{k}\sigma}], \end{aligned} \quad (21)$$

where the sum of momenta \mathbf{k} is performed over the reduced Brillouin zone (RBZ) with AF ordering wave vector \mathbf{Q} , with $\Lambda_A = -2JA\gamma_{\mathbf{k}}$, $\Lambda_B = -4tB\gamma_{\mathbf{k}}$, with $\gamma_{\mathbf{k}} = \frac{1}{Z} \sum_{\vec{\delta}} e^{i\mathbf{k} \cdot \vec{\delta}}$ ($\vec{\delta}$ are nearest-neighbor vectors). Note that when $B = 0$ in H_{MF} , the $SU(2)$ spin and $U(1)$, \mathcal{S}^{00} , symmetries are conserved; the opposite case $A = 0$ preserves $\mathcal{S}^{10(01)} + \mathcal{S}^{20(02)}$ and $\mathcal{S}^{11} + \mathcal{S}^{22} - \mathcal{S}^{00}$ symmetries. In Eq. (21) we have only considered homogeneous solutions [14].

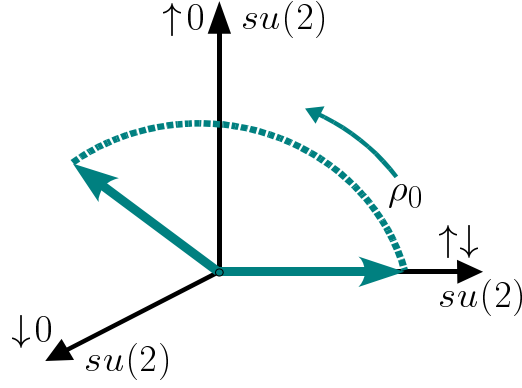


Figure 2. Schematics of the order parameter change (as a function of ρ_0) with the three $su(2)$ axis describing 3 different $su(2)$ subalgebras of $su(3)$. Note that the plane $\uparrow 0 - \downarrow 0$ represents the 5 charge symmetry generators while the $\uparrow\downarrow$ axis is associated to the remaining three generators of magnetism.

The corresponding self-consistent MF equations to solve are

$$\begin{cases} A = \frac{8}{zN_s} \sum_{\mathbf{k} \in \text{RBZ}} \gamma_{\mathbf{k}} \langle b_{\mathbf{k}\uparrow}^\dagger b_{-\mathbf{k}+\mathbf{Q}\downarrow}^\dagger \rangle_{MF} , \\ B = \frac{8}{zN_s} \sum_{\mathbf{k} \in \text{RBZ}} \gamma_{\mathbf{k}} \langle b_{\mathbf{k}\sigma}^\dagger b_{-\mathbf{k}+\mathbf{Q}0}^\dagger \rangle_{MF} , \\ 1 = \frac{1}{N_s} \sum_{\mathbf{k} \in \text{RBZ}} \sum_{\alpha} \langle n_{\mathbf{k}\alpha} \rangle_{MF} . \end{cases} \quad (22)$$

We are thus left with a non-interacting system of SW bosons. Now we follow Colpa [15] and diagonalize para-unitarily the Hamiltonian matrix \tilde{H} . The application of a homogeneous linear transformation leads to [10]

$$\tilde{H} = \sum_{\mathbf{k} \in \text{RBZ}} \sum_{i=0}^5 \omega_{i\mathbf{k}} \alpha_{i\mathbf{k}}^\dagger \alpha_{i\mathbf{k}} , \quad (23)$$

where the mode energies $\omega_{i\mathbf{k}}$ are, at least, two-fold degenerate. In Fig. 1, we display the orders A and B as a function of $\rho_0 = \frac{1}{N_s} \sum_{\mathbf{j}} \langle n_{\mathbf{j}0} \rangle$ at very low temperature and different J/t ratios for a two-dimensional lattice [16]. The relation between the OPs of the original problem, Eq. (17), and A and B is given by $\frac{1}{N_s} \sum_{\mathbf{i}, \mathbf{j}} e^{i\mathbf{Q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \langle \bar{b}_{i\sigma}^\dagger \bar{b}_{j\sigma'} \rangle_{MF} \propto B^2$, $\frac{1}{N_s} \sum_{\mathbf{i}, \mathbf{j}} e^{i\mathbf{Q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \langle s_i^+ s_j^- \rangle_{MF} \propto A^2$, and justifies the labeling of the phases AF (antiferromagnet) and BEC (Bose-Einstein condensate) in Fig. 1.

A way to qualitatively understand this quantum phase diagram is to look into the OP space as displayed in Fig. 2. As ρ_0 (or chemical potential) varies from 0 to 1, the OP (depicted as an arrow) moves in OP space. When $\rho_0 = 0$ the order is purely AF and the arrow lies on the $\uparrow\downarrow$ axis. For $\rho_0 \neq 0$, the arrow has projections onto the 3 $su(2)$ axis, i.e., the AF state *coexists* with a BEC state. There is a particular critical value of $\rho_0 = \rho_{0c} < 1$ for which the AF ordering vanishes and the OP is

purely BEC with the arrow lying in the $\uparrow 0 - \downarrow 0$ plane. This BEC ordering persists until $\rho_0 = 1$, where it vanishes.

There are some open issues. One regards the application of the HMFT approach to study fermionic problems, for example, a Hamiltonian like Eq. (17) but where the operators $\bar{b}_{i\sigma}^\dagger$ for different modes on a lattice \mathbf{i} anticommute. The method could certainly be used, however, fermions do introduce a non-local gauge potential [9] leading to an effective dynamical frustration which is difficult to handle in a controlled manner. Another issue concerns the application of the HMFT method when longer-range interactions are involved. There is already evidence from work on the J_1 - J_2 $SU(2)$ Heisenberg model [17] that our HMFT will work in those cases as far as homogeneous phases are concerned. Actually, the SW MF theory introduced by Arovas and Auerbach [18] in the fundamental representation is a particular case of our general HMFT. Finally, our methodology exhausts all broken symmetry instances but it is still quite possible to have purely topological quantum orders and their corresponding phase transitions which cannot be described by broken symmetries and associated OPs [19] and, thus, are not included in our framework.

3. SUMMARY

Summarizing, we developed a theoretical framework and a calculational scheme to study coexistence and competition of thermodynamic phases in strongly correlated matter. In our method (given a Hamiltonian modeling the physical system) the order parameters are not guessed but rigorously determined from group theoretical considerations as symmetry generators of a hierarchical language. In this way, the Hamiltonian operator (which does not necessarily have the full symmetry of the hierarchical group) is expressed in terms of symmetry generators. Then, in a non-phenomenological approach dubbed *hierarchical mean-field theory*, we approximated the dynamics (and thermodynamics) treating all possible local order parameters on an equal footing, i.e., without preferred symmetry axis. One could say that this procedure follows the guiding principles of *maximum symmetry* and *minimum information*. This allowed us to obtain in a simple manner the phase diagram of a model problem exhibiting a superfluid-insulator quantum phase transition and another displaying coexistence and competition between antiferromagnetism and superfluidity. Combined with an analysis of the fluctuations (to analyze the stability of the mean-field) one now has a simple machinery to design phase diagrams.

As can be inferred from our presentation, there are two complementary aspects to studying competition and coexistence between phase orderings in strongly coupled quantum systems. One is the direct discovery of the *hidden unity* and subsequent determination of the possible phases and their transitions, given a Hamiltonian operator modeling the complex material of interest. This is the aspect we have described in the present paper. The second aspect, to be discussed in a separate publication, involves the design or engineering of new states of matter (i.e., new quantum orders) using the inverse path of logic. Essentially, the idea consists of tailoring effective Hamiltonians based upon a general symmetry analysis of the possible orderings one would like to realize at zero temperature. Tuning the parameters of these symmetry-based effective Hamiltonians allows one to move in parameter space along the previously established orderings. Indeed, this strategy finds its experimental realization in recent

work done on atomic BEC systems in optical lattices [20], and our approach provides a unique theoretical guidance to achieve that goal given the possibility of control and tunability of the interactions of the elementary constituents (i.e, quantum control).

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- [10] Irrelevant constant terms in the transformed Hamiltonians are always omitted.
- [11] Formally, a language is defined by a set of operators each acting irreducibly on a local Hilbert space of dimension D . The language does not determine the quantum dynamics of the system (which is determined by the Hamiltonian) but the dynamics can be expressed in different languages. See Refs. [4,5].
- [12] We know that for $J = V = 2t = \frac{2\bar{\mu}}{Z} < 0$, we can find the exact ground and lowest energy states of H [3].
- [14] If we are looking for solutions which could break the lattice translational symmetry (inhomogeneous), we should consider OPs $\mathcal{S}^{\mu\nu}(\mathbf{k}) = \frac{1}{N_s} \sum_{\mathbf{j}} e^{i\mathbf{k}\cdot\mathbf{r}_{\mathbf{j}}} \mathcal{S}^{\mu\nu}(\mathbf{j})$, for all possible \mathbf{k} -vectors. In general, non-local OPs can be derived (starting from our local classification) similarly to what has been done in the literature. For instance, for a $S = 1$ system there are two possible local OPs: the local magnetization and spin-nematic parameters. Thus, in the case of a quantum spin ($S = 1$) glass there will be two types of glasses: One where the local magnetization is frozen at each site and another where the local spin-nematic order is frozen at each site. In other words, there will be two Edwards-Anderson OPs.
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